Enumeration: Wrap Up

Recall: What do we know?

① **Addition Principle:** $\#(A \cup B) = (\#A) + (\#B)$ when $A \cap B = \emptyset$.

② **Product Principle:** $\#(A \times B) = (\#A) \cdot (\#B)$.

- Permutations of $[n]$: $n!$
- k-sequences in $[n]$: n^k
- Subsets of $[n]$: 2^n
- Ordered k-subsets of $[n]$: $\frac{n!}{(n-k)!}$
- k-subsets of $[n]$: $\binom{n}{k} \overset{\text{Thm}}{=} \frac{n!}{k!\,(n-k)!}$
- Various identities on binomial coefficients...
- Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

③ "Counting in Two Ways."

Principle of Inclusion-Exclusion

Obs: In addition principle, we needed sets to be disjoint...

$\#(A \cup B) \neq (\#A) + (\#B)$ when $A \cap B \neq \emptyset$.

Ex: Let $A = \{1,2,3,4\}$, $B = \{1,3,5,6\}$:

$\#A = 4 = \#B$.

$A \cup B = \{1,2,3,4,5,6\} = [6]$, so $\#(A \cup B) = 6 \neq 4+4 \ldots$

$A \cap B = \{1,3\}$, so $\#(A \cap B) = 2$.

$\#(A \cup B) = 4+4-2 = (\#A) + (\#B) - (\#(A \cap B))$...

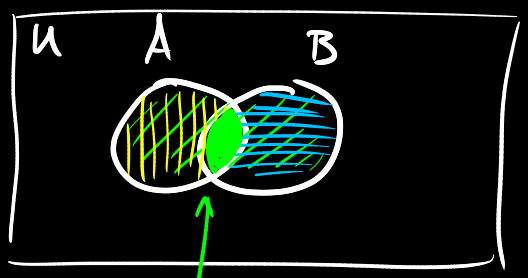
Prop (Inclusion-Exclusion Principle): Given finite sets A and B,

$$\#(A \cup B) + \#(A \cap B) = (\#A) + (\#B).$$

In particular, $$\#(A \cup B) = (\#A) + (\#B) - (\#(A \cap B)).$$

IDEA: Inclusion-Exclusion corrects the overcounting done by $(\#A)+(\#B)$ via subtraction.

Picture:



$(\#A)+(\#B)$ counts $A \cap B$ twice...

Observation: when $A \cap B = \emptyset$, this formula says

$$\begin{aligned}\#(A \cup B) &= (\#A) + (\#B) - (\#(A \cap B)) \\ &= (\#A) + (\#B) - \#\emptyset \\ &= (\#A) + (\#B).\end{aligned}$$

Thus Inclusion-Exclusion principle generalizes Addition Principle.

pf (Inc-Ex): Let A and B be arbitrary finite sets.

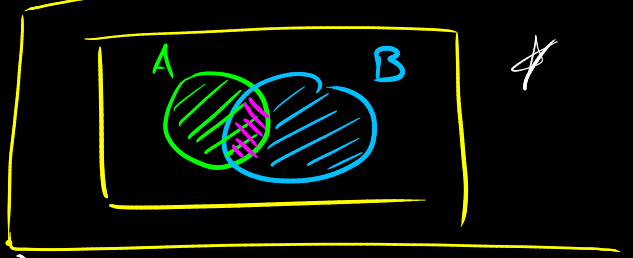
Consider $C = (A \times \{1\}) \cup (B \times \{2\})$. By the Product Principle, $\#(A \times \{1\}) = (\#A) \cdot 1$ and $\#(B \times \{2\}) = (\#B) \cdot 1$.

Moreover, $(A \times \{1\}) \cap (B \times \{2\}) = (A \cap B) \times (\{1\} \cap \{2\}) = \emptyset$.

Hence, $\#C = (\#A) + (\#B)$. On the other hand, we can rewrite $C = [(A \times \{1\}) \cup ((B \setminus A) \times \{2\})] \cup ((A \cap B) \times \{2\})$.

Now $\#[(A \times \{1\}) \cup ((B \setminus A) \times \{2\})]$

$= \#(A \cup B)$ by the Correspondence principle $\left(\begin{matrix} a \in A \longleftrightarrow (a,1) \\ b \in B \setminus A \longleftrightarrow (b,2)\end{matrix}\right)$



Moreover, $\#((A \cap B) \times \{2\}) = \#(A \cap B) \cdot 1$ by the Product Principle.

Hence $\#(A \cup B) + \#(A \cap B) = \#C = (\#A) + (\#B)$ as desired. ☒

Ex: A classroom has 21 students who like coffee and 37 students who like chocolate. Every student likes either coffee or chocolate, and 12 like both. How many students are there?

Sol: Let A and B denote the set of students who like coffee and chocolate respectively. We are told:

① The full set of students is $A \cup B$

② $\#A = 21$, $\#B = 37$, and $\#(A \cap B) = 12$.

Hence $\#(A \cup B) = (\#A) + (\#B) - (\#(A \cap B))$ (by Inclusion-Exclusion)

$= 21 + 37 - 12 = 46$,

and there are thus 46 students in the class. ☒

Remark: There is a much more general formulation for Inclusion-Exclusion... As an exercise:

Exercise: ① Find a formula for $\#(A \cup B \cup C)$ using sizes of A, B, C, their pairwise intersections, and $\#(A \cap B \cap C)$.

↳ **Big hint:** Draw a picture to start....

② Prove it (using inc-exc. formula).

③ Conjecture and prove a similar formula for $\#(A_1 \cup A_2 \cup \cdots \cup A_n)$.

Pigeonhole Principle

Observation: Given n containers and $n+1$ objects, every distribution of the objects into the containers must put some two of the objects into the same container.

↑ that's the Pigeonhole principle... Proof is left to you.

Ex: Given any 7 integers, some two are equivalent modulo 6. Indeed, make one box for each remainder modulo 6. There are 6 such boxes. Sort the 7 integers into these boxes. By the Pigeonhole Principle, some two of those 7 are in the same box. Thus, those two (at least) have the same remainder modulo 6. ☑

Ex: There are at least 2 people in New York City with the same number of hairs on their heads. Indeed, empirical observations have shown the human head has no more than 1 million hairs. There are more (by quite a bit) than 4 million people living in NYC. Choose any 1,000,001 of these, and sort them into boxes labeled by number of hairs on their head. Thus some two people were sorted into the same box by the Pigeonhole-Principle, so those two people (at least) have the same number of hairs on their heads. ☑

Now let's strengthen/generalize the Pigeonhole Principle !!

Defn: A weak partition of set S is a collection $\mathcal{F} \subseteq \mathbb{P}(S)$ such that $\bigcup_{A \in \mathcal{F}} A = S$.

(i.e. $\mathcal{F}$ "covers" all of S). A weak partition is a partition when its elements are disjoint pairwise.

Point: "Boxes" are elements of a weak partition.

Recall: The ceiling of real number x is

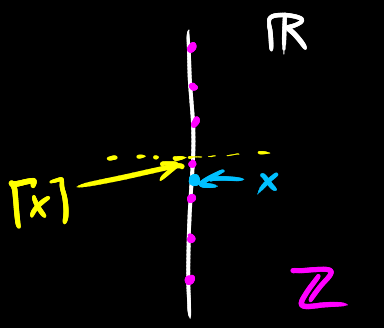
$$\lceil x \rceil = \min \{ n \in \mathbb{Z} : x \leq n \}.$$

i.e. $\lceil x \rceil$ is the smallest integer which is at least x.

Ex: $\lceil \pi \rceil = 4$,

$\lceil e \rceil = 3$, $\lceil \frac{1}{2} \rceil = 1$

Picture:



Prop (Generalized Pigeonhole Principle): If $\mathcal{F}$ is a weak partition of finite set S into r parts, then there are at least $\lceil \frac{\#S}{r} \rceil$ elements in some part.

Ex: If $\#S = n+1$ and $\#\mathcal{F} = n$, then $\lceil \frac{n+1}{n} \rceil = \lceil 1 + \frac{1}{n} \rceil = 2$,

so we recover the Pigeonhole principle...

Ex: On hold ('til next time).

$\rightsquigarrow$ we'll see this is stronger than Pigeonhole Principle...